

# Homogenization in Perforated Domains and Interior Lipschitz Estimates

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## Abstract

We establish interior Lipschitz estimates at the macroscopic scale for solutions to systems of linear elasticity with rapidly oscillating periodic coefficients and mixed boundary conditions in domains periodically perforated at a microscopic scale  $\varepsilon$  by establishing  $H^1$ -convergence rates for such solutions. The interior estimates are derived directly without the use of compactness via an argument presented in [3] that was adapted for elliptic equations in [2] and [11]. As a consequence, we derive a Liouville type estimate for solutions to the systems of linear elasticity in unbounded periodically perforated domains.

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*Keywords:* Homogenization; Linear elasticity; Elliptic systems; Lipschitz estimates

## 1 Introduction

The purpose of this paper is to establish  $H^1$ -convergence rates in periodic homogenization and to establish interior Lipschitz estimates at the macroscopic scale for solutions to systems of linear elasticity in domains periodically perforated at a microscopic scale  $\varepsilon$ . To be precise, we consider the operator

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A^\varepsilon(x)\nabla) = -\frac{\partial}{\partial x_i} \left( a_{ij}^{\alpha\beta} \left( \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right), \quad x \in \varepsilon\omega, \varepsilon > 0, \quad (1.1)$$

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where  $A^\varepsilon(x) = A(x/\varepsilon)$ ,  $A(y) = \{a_{ij}^{\alpha\beta}(y)\}_{1 \leq i,j,\alpha,\beta \leq d}$  for  $y \in \omega$ ,  $d \geq 2$ , and  $\omega \subseteq \mathbb{R}^d$  is an unbounded Lipschitz domain with 1-periodic structure, i.e., if  $\mathbf{1}_+$  denotes the characteristic function of  $\omega$ , then  $\mathbf{1}_+$  is a 1-periodic function in the sense that

$$\mathbf{1}_+(y) = \mathbf{1}_+(z + y) \quad \text{for } y \in \mathbb{R}^d, z \in \mathbb{Z}^d.$$

The summation convention is used throughout. We write  $\varepsilon\omega$  to denote the  $\varepsilon$ -homothetic set  $\{x \in \mathbb{R}^d : x/\varepsilon \in \omega\}$ . We assume  $\omega$  is connected and that any two connected components of  $\mathbb{R}^d \setminus \omega$  are separated by some positive distance. This is stated more precisely in Section 2. We also assume each connected component of  $\mathbb{R}^d \setminus \omega$  is bounded.

We assume the coefficient matrix  $A(y)$  is real, measurable, and satisfies the elasticity conditions

$$a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \quad (1.2)$$

$$\kappa_1 |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \kappa_2 |\xi|^2, \quad (1.3)$$

for  $y \in \omega$  and any symmetric matrix  $\xi = \{\xi_i^\alpha\}_{1 \leq i,\alpha \leq d}$ , where  $\kappa_1, \kappa_2 > 0$ . We also assume  $A$  is 1-periodic, i.e.,

$$A(y) = A(y + z) \quad \text{for } y \in \omega, z \in \mathbb{Z}^d. \quad (1.4)$$

The coefficient matrix of the systems of linear elasticity describes the linear relation between the stress and strain a material experiences during relatively small elastic deformations. Consequently, the elasticity conditions (1.2) and (1.3) should be regarded as physical parameters of the system, whereas  $\varepsilon$  is clearly a geometric parameter.

For a bounded domain  $\Omega \subset \mathbb{R}^d$ , we write  $\Omega_\varepsilon$  to denote the domain  $\Omega_\varepsilon = \Omega \cap \varepsilon\omega$ . In this paper, we consider the mixed boundary value problem given by

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ \sigma_\varepsilon(u_\varepsilon) = 0 & \text{on } S_\varepsilon := \partial\Omega_\varepsilon \cap \Omega \\ u_\varepsilon = f & \text{on } \Gamma_\varepsilon := \partial\Omega_\varepsilon \cap \partial\Omega, \end{cases} \quad (1.5)$$

where  $\sigma_\varepsilon = -nA^\varepsilon(x)\nabla$  and  $n$  denotes the outward unit normal to  $\Omega_\varepsilon$ . We say  $u_\varepsilon$  is a weak solution to (1.5) provided

$$\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta\varepsilon} \frac{\partial u_\varepsilon^\beta}{\partial x_j} \frac{\partial w^\alpha}{\partial x_i} = 0, \quad w = \{w^\alpha\}_\alpha \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d), \quad (1.6)$$

and  $u_\varepsilon - f \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ , where  $H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$  denotes the closure in  $H^1(\Omega_\varepsilon; \mathbb{R}^d)$  of  $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$  functions vanishing on  $\Gamma_\varepsilon$ . The boundary value problem (1.5) models relatively small elastic deformations of composite materials subject to zero external body forces (see [8]).

If  $\omega = \mathbb{R}^d$ —the case when  $\Omega_\varepsilon = \Omega$ —then the existence and uniqueness of a weak solution  $u_\varepsilon \in H^1(\Omega_\varepsilon; \mathbb{R}^d)$  to (1.5) for a given  $f \in H^1(\Omega; \mathbb{R}^d)$  follows easily from the Lax-Milgram theorem and Korn's first inequality. If  $\omega \subsetneq \mathbb{R}^d$ , then the existence and uniqueness of a weak solution to (1.5) still follows from the Lax-Milgram theorem but in addition Korn's first inequality for perforated domains (see Lemma 2.6).

One of the main results of this paper is the following theorem. For any measurable set  $E$  (possibly empty) and ball  $B(x_0, r) \subset \mathbb{R}^d$  with  $r > 0$ , denote

$$\int_{B(x_0, r) \cap E} f(x) dx = \frac{1}{r^d} \int_{B(x_0, r) \cap E} f(x) dx$$

**Theorem 1.1.** *Suppose  $A$  satisfies (1.2), (1.3), and (1.4). Let  $u_\varepsilon$  denote a weak solution to  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B(x_0, R) \cap \varepsilon\omega$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  for  $B(x_0, R) \cap \partial(\varepsilon\omega)$  for some  $x_0 \in \mathbb{R}^d$  and  $R > 0$ . For  $\varepsilon \leq r < R/3$ , there exists a constant  $C$  depending on  $d$ ,  $\omega$ ,  $\kappa_1$ , and  $\kappa_2$  such that*

$$\left( \int_{B(x_0, r) \cap \varepsilon\omega} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left( \int_{B(x_0, R) \cap \varepsilon\omega} |\nabla u_\varepsilon|^2 \right)^{1/2}. \quad (1.7)$$

The scale-invariant estimate in Theorem 1.1 should be regarded as a Lipschitz estimate for solutions  $u_\varepsilon$ , as under additional smoothness assumptions on the coefficients  $A$  we may deduce interior Lipschitz estimate for solutions to (1.5) from local Lipschitz estimates for  $\mathcal{L}_1$  and a “blow-up argument” (see the proof of Lemma 4.2). In particular, if  $A$  is Hölder continuous, i.e., there exists a  $\tau \in (0, 1)$  with

$$|A(x) - A(y)| \leq C|x - y|^\tau \quad \text{for } x, y \in \omega \quad (1.8)$$

for some constant  $C$  uniform in  $x$  and  $y$ , we may deduce the following corollary.

**Corollary 1.2.** *Suppose  $A$  satisfies (1.2), (1.3), (1.4), and (1.8), and suppose  $\omega$  is an unbounded  $C^{1, \alpha}$  domain for some  $\alpha > 0$ . Let  $u_\varepsilon$  denote a weak solution to  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B(x_0, R) \cap \varepsilon\omega$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  for  $B(x_0, R) \cap \partial(\varepsilon\omega)$  for some  $x_0 \in \mathbb{R}^d$  and  $R > 0$ . Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(B(x_0, R/3) \cap \varepsilon\omega)} \leq C \left( \int_{B(x_0, R) \cap \varepsilon\omega} |\nabla u_\varepsilon|^2 \right)^{1/2}, \quad (1.9)$$

where  $C$  depends on  $d$ ,  $\omega$ ,  $\kappa_1$ ,  $\kappa_2$ ,  $\tau$ , and  $\alpha$ .

Another consequence of Theorem 1.1 is the following Liouville type property for systems of linear elasticity in unbounded periodically perforated domains. In particular, we have the following corollary.

**Corollary 1.3.** *Suppose  $A$  satisfies (1.2), (1.3), and (1.4), and suppose  $\omega$  is an unbounded Lipschitz domain with 1-periodic structure. Let  $u$  denote a weak solution of  $\mathcal{L}_1(u) = 0$  in  $\omega$  and  $\sigma_1(u) = 0$  on  $\partial\omega$ . Assume*

$$\left( \int_{B(0,R) \cap \omega} |u|^2 \right)^{1/2} \leq CR^\nu, \quad (1.10)$$

for some  $\nu \in (0, 1)$ , some constant  $C := C(u) > 0$ , and for all  $R > 1$ . Then  $u$  is constant.

Interior Lipschitz estimates for the case  $\omega = \mathbb{R}^d$  were first obtained *indirectly* through the method of compactness presented in [4]. Interior Lipschitz estimates for solutions to a single elliptic equation in the case  $\omega \subsetneq \mathbb{R}^d$  were obtained indirectly in [?] through the same method of compactness. The method of compactness is essentially a “proof by contradiction” and relies on the qualitative convergence of solutions  $u_\varepsilon$  (see Theorem 2.7). The method relies on sequences of operators  $\{\mathcal{L}_{\varepsilon_k}^k\}_k$  and sequences of functions  $\{u_k\}_k$  satisfying  $\mathcal{L}_{\varepsilon_k}^k(u_k) = 0$ , where  $\mathcal{L}_{\varepsilon_k}^k = -\operatorname{div}(A_k^{\varepsilon_k} \nabla)$ ,  $\{A_k^{\varepsilon_k}\}_k$  satisfies (1.2), (1.3), and (1.4) in  $\omega + s_k$  for  $s_k \in \mathbb{R}^d$ . In the case  $\omega = \mathbb{R}^d$ , then  $\omega + s_k = \mathbb{R}^d$  for any  $s_k \in \mathbb{R}^d$ , and so it is clear that estimate (1.7) is uniform in affine transformations of  $\omega$ . In the case  $\omega \subsetneq \mathbb{R}^d$ , affine shifts of  $\omega$  must be considered, which complicates the general scheme.

Interior Lipschitz estimates for the case  $\omega = \mathbb{R}^d$  were obtained *directly* in [11] through a general scheme for establishing Lipschitz estimates at the macroscopic scale first presented in [3] and then modified for second-order elliptic systems in [2] and [11]. We emphasize that our result is unique in that Theorem 1.1 extends estimates presented in [11]—i.e., interior Lipschitz estimates for systems of linear elasticity—to the case  $\omega \subsetneq \mathbb{R}^d$  while *completely avoiding the use of compactness methods*.

The proof of Theorem 1.1 (see Section 4) relies on the quantitative convergence rates of the solutions  $u_\varepsilon$ . Let  $u_0 \in H^1(\Omega; \mathbb{R}^d)$  denote the weak solution of the boundary value problem for the homogenized system corresponding to (1.5) (see (2.6)), and let  $\chi =$

$\{\chi_j^\beta\}_{1 \leq j, \beta \leq d} \in H_{\text{per}}^1(\omega; \mathbb{R}^d)$  denote the matrix of correctors (see (2.8)), where  $H_{\text{per}}^1(\omega; \mathbb{R}^d)$  denotes the closure in  $H^1(Q \cap \omega; \mathbb{R}^d)$  of the set of 1-periodic  $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$  functions and  $Q = [-1/2, 1/2]^d$ . In the case  $\omega \subsetneq \mathbb{R}^d$ , the estimate

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon \nabla u_0\|_{H^1(\Omega_\varepsilon)} \leq C \varepsilon^{1/2} \|u_0\|_{H^3(\Omega)}$$

was proved in [10] under the assumption that  $\chi_j^\beta \in W_{\text{per}}^{1,\infty}(\omega; \mathbb{R}^d)$  for  $1 \leq j, \beta \leq d$ , where  $W_{\text{per}}^{1,\infty}(\omega; \mathbb{R}^d)$  is defined similarly to  $H_{\text{per}}^1(\omega; \mathbb{R}^d) = W_{\text{per}}^{1,2}(\omega; \mathbb{R}^d)$ . However, if it is only assumed that the coefficients  $A$  are real, measurable, and satisfy (1.2), (1.3), and (1.4), then the first-order correctors are not necessarily Lipschitz. Consequently, the following theorem is another main result of this paper. Let  $K_\varepsilon$  denote the smoothing operator at scale  $\varepsilon$  defined by (2.1), and let  $\eta_\varepsilon \in C_0^\infty(\Omega)$  be the cut-off function defined by (3.1). The use of the smoothing operator  $K_\varepsilon$  (details are discussed in Section 2) is motivated by work in [12].

**Theorem 1.4.** *Let  $\Omega$  be a bounded Lipschitz domain and  $\omega$  be an unbounded Lipschitz domain with 1-periodic structure. Suppose  $A$  is real, measurable, and satisfies (1.2), (1.3), and (1.4). Let  $u_\varepsilon$  denote a weak solution to (1.5). There exists a constant  $C$  depending on  $d$ ,  $\Omega$ ,  $\omega$ ,  $\kappa_1$ , and  $\kappa_2$  such that*

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_0) \eta_\varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)}.$$

This paper is structured in the following manner. In Section 2, we establish notation and recall various preliminary results from other works. The convergence rate presented in Theorem 1.4 is proved in Section 3. In Section 4, we prove the interior Lipschitz estimates given by Theorem 1.1 and provide the proof of Corollary 1.2. To finish the section, we prove the Liouville type property Corollary 1.3.

## 2 Notation and Preliminaries

Fix  $\zeta \in C_0^\infty(B(0, 1))$  so that  $\zeta \geq 0$  and  $\int_{\mathbb{R}^d} \zeta = 1$ . Define

$$K_\varepsilon(g)(x) = \int_{\mathbb{R}^d} g(x - y) \zeta_\varepsilon(y) dy, \quad f \in L^2(\mathbb{R}^d) \quad (2.1)$$

where  $\zeta_\varepsilon(y) = \varepsilon^{-d} \zeta(y/\varepsilon)$ . Note  $K_\varepsilon$  is a continuous map from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^d)$ . A proof for each of the following two lemmas is readily

available in [11], and so we do not present either here. For any function  $g$ , set  $g^\varepsilon(\cdot) = g(\cdot/\varepsilon)$ .

**Lemma 2.1.** *Let  $g \in H^1(\mathbb{R}^d)$ . Then*

$$\|g - K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon \|\nabla g\|_{L^2(\mathbb{R}^d)},$$

where  $C$  depends only on  $d$ .

**Lemma 2.2.** *Let  $h \in L^2_{loc}(\mathbb{R}^d)$  be a 1-periodic function. Then for any  $g \in L^2(\mathbb{R}^d)$ ,*

$$\|h^\varepsilon K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C \|h\|_{L^2(Q)} \|g\|_{L^2(\mathbb{R}^d)}$$

A proof of Lemma 2.3 can be found in [10].

**Lemma 2.3.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. For any  $g \in H^1(\Omega)$ ,*

$$\|g\|_{L^2(\mathcal{O}_r)} \leq Cr^{1/2} \|g\|_{H^1(\Omega)},$$

where  $C$  depends on  $d$  and  $\Omega$ , and  $\mathcal{O}_r = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}$ .

A proof of Lemma 2.4 can be found in [8].

**Lemma 2.4.** *Suppose  $B = \{b_{ij}^{\alpha\beta}\}_{1 \leq i,j,\alpha,\beta \leq d}$  is 1-periodic and satisfies  $b_{ij}^{\alpha\beta} \in L^2_{loc}(\mathbb{R}^d)$  with*

$$\frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} = 0, \quad \text{and} \quad \int_Q b_{ij}^{\alpha\beta} = 0.$$

*There exists  $\pi = \{\pi_{kij}^{\alpha\beta}\}_{1 \leq i,j,k,\alpha,\beta \leq d}$  with  $\pi_{kij}^{\alpha\beta} \in H^1_{loc}(\mathbb{R}^d)$  that is 1-periodic and satisfies*

$$\frac{\partial}{\partial y_k} \pi_{kij}^{\alpha\beta} = b_{ij}^{\alpha\beta} \quad \text{and} \quad \pi_{kij}^{\alpha\beta} = -\pi_{ikj}^{\alpha\beta}.$$

Theorem 2.5 is a classical result in the study of periodically perforated domains. It can be used to prove Korn's first inequality in perforated domains (see Lemma 2.6), which is needed together with the Lax-Milgram theorem to prove the existence and uniqueness of solutions to (1.5). For a proof of Theorem 2.5, see [10].

**Theorem 2.5.** *Let  $\Omega$  and  $\Omega_0$  be a bounded Lipschitz domains with  $\overline{\Omega} \subset \Omega_0$  and  $\text{dist}(\partial\Omega_0, \Omega) > 1$ . For  $0 < \varepsilon < 1$ , there exists a linear extension operator  $P_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \rightarrow H_0^1(\Omega_0; \mathbb{R}^d)$  such that*

$$\|P_\varepsilon w\|_{H^1(\Omega_0)} \leq C_1 \|w\|_{H^1(\Omega_\varepsilon)}, \quad (2.2)$$

$$\|\nabla P_\varepsilon w\|_{L^2(\Omega_0)} \leq C_2 \|\nabla w\|_{L^2(\Omega_\varepsilon)}, \quad (2.3)$$

$$\|e(P_\varepsilon w)\|_{L^2(\Omega_0)} \leq C_3 \|e(w)\|_{L^2(\Omega_\varepsilon)}, \quad (2.4)$$

for some constants  $C_1$ ,  $C_2$ , and  $C_3$  depending on  $\Omega$  and  $\omega$ , where  $e(w)$  denotes the symmetric part of  $\nabla w$ , i.e.,

$$e(w) = \frac{1}{2} [\nabla w + (\nabla w)^T]. \quad (2.5)$$

Korn's inequalities are classical in the study of linear elasticity. The following lemma is essentially Korn's first inequality but formatted for periodically perforated domains. Lemma 2.6 follows from Theorem 2.5 and Korn's first inequality. For an explicit proof of Lemma 2.6, see [10].

**Lemma 2.6.** *There exists a constant  $C$  independent of  $\varepsilon$  such that*

$$\|w\|_{H^1(\Omega_\varepsilon)} \leq C \|e(w)\|_{L^2(\Omega_\varepsilon)}$$

for any  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ , where  $e(w)$  is given by (2.5).

If  $\omega = \mathbb{R}^d$ , it can be shown that the weak solution to (1.5) converges weakly in  $H^1(\Omega; \mathbb{R}^d)$  and consequently strongly in  $L^2(\Omega; \mathbb{R}^d)$  as  $\varepsilon \rightarrow 0$  to some  $u_0$ , which is a solution of a boundary value problem in the domain  $\Omega$  (see [5] or [8]). Indeed, we have the following known qualitative convergence.

**Theorem 2.7.** *Suppose  $\omega = \mathbb{R}^d$  and that  $\Omega$  is a bounded Lipschitz domain. Suppose  $A$  satisfies (1.2), (1.3), and (1.4). Let  $u_\varepsilon$  satisfy  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $\Omega$ , and  $u_\varepsilon = f$  on  $\partial\Omega$ . Then there exists a  $u_0 \in H^1(\Omega; \mathbb{R}^d)$  such that*

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d).$$

Consequently,  $u_\varepsilon \rightarrow u_0$  strongly in  $L^2(\Omega; \mathbb{R}^d)$ .

For a proof of the previous theorem, see [5], Section 10.3. The function  $u_0$  is called the homogenized solution and the boundary value problem it solves is the homogenized system corresponding to (1.5).

If  $\omega \subsetneq \mathbb{R}^d$ , then it is difficult to qualitatively discuss the convergence of  $u_\varepsilon$ , as  $H^1(\Omega_\varepsilon; \mathbb{R}^d)$  and  $L^2(\Omega_\varepsilon; \mathbb{R}^d)$  depend explicitly on  $\varepsilon$ . Qualitative convergence in this case is discussed in [1], [6], and others. The homogenized system of elasticity corresponding to (1.5) and of which  $u_0$  is a solution is given by

$$\begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } \Omega \\ u_0 = f & \text{on } \partial\Omega, \end{cases} \quad (2.6)$$

where  $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$ ,  $\hat{A} = \{\hat{a}_{ij}^{\alpha\beta}\}_{1 \leq i,j,\alpha,\beta \leq d}$  denotes a constant matrix given by

$$\hat{a}_{ij}^{\alpha\beta} = \int_{Q \cap \omega} a_{ik}^{\alpha\gamma} \frac{\partial \mathbb{X}_j^{\gamma\beta}}{\partial y_k}, \quad (2.7)$$

and  $\mathbb{X}_j^\beta = \{\mathbb{X}_j^{\gamma\beta}\}_{1 \leq \gamma \leq d}$  denotes the weak solution to the boundary value problem

$$\begin{cases} \mathcal{L}_1(\mathbb{X}_j^\beta) = 0 & \text{in } Q \cap \omega \\ \sigma_1(\mathbb{X}_j^\beta) = 0 & \text{on } \partial\omega \cap Q \\ \chi_j^\beta := \mathbb{X}_j^\beta - y_j e^\beta \text{ is 1-periodic, } \int_{Q \cap \omega} \chi_j^\beta = 0, \end{cases} \quad (2.8)$$

where  $e^\beta \in \mathbb{R}^d$  has a 1 in the  $\beta$ th position and 0 in the remaining positions. For details on the existence of solutions to (2.8), see [10]. The functions  $\chi_j^\beta$  are referred to as the first-order correctors for the system (1.5).

It is assumed that any two connected components of  $\mathbb{R}^d \setminus \omega$  are separated by some positive distance. Specifically, if  $\mathbb{R}^d \setminus \omega = \cup_{k=1}^\infty H_k$  where  $H_k$  is connected and bounded for each  $k$ , then there exists a constant  $\mathfrak{g}^\omega$  so that

$$0 < \mathfrak{g}^\omega \leq \inf_{i \neq j} \left\{ \inf_{\substack{x_i \in H_i \\ x_j \in H_j}} |x_i - x_j| \right\}. \quad (2.9)$$

### 3 Convergence Rates in $H^1(\Omega_\varepsilon)$

In this section, we establish  $H^1(\Omega_\varepsilon)$ -convergence rates for solutions to (1.5) by proving Theorem 1.4. It should be noted that if  $A$  satisfies (1.2) and (1.3), then  $\hat{A}$  defined by (2.7) satisfies conditions (1.2)



and (1.3) but with possibly different constants  $\widehat{\kappa}_1$  and  $\widehat{\kappa}_2$  depending on  $\kappa_1$  and  $\kappa_2$ . In particular, we have the following lemma. For a proof of Lemma 3.1, see either [5], [8], or [10].

**Lemma 3.1.** *Suppose  $A$  satisfies (1.2), (1.3), and (1.4). If  $\mathbb{X}_j^\beta = \{\mathbb{X}_j^{\gamma\beta}\}_\gamma$  denote the weak solutions to (2.8), then  $\widehat{A} = \{\widehat{a}_{ij}^{\alpha\beta}\}$  defined by*

$$\widehat{a}_{ij}^{\alpha\beta} = \int_{Q \cap \omega} a_{ik}^{\alpha\gamma} \frac{\partial \mathbb{X}_j^{\gamma\beta}}{\partial y_k}$$

*satisfies  $\widehat{a}_{ij}^{\alpha\beta} = \widehat{a}_{ji}^{\beta\alpha} = \widehat{a}_{\alpha j}^{i\beta}$  and*

$$\widehat{\kappa}_1 |\xi|^2 \leq \widehat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \leq \widehat{\kappa}_2 |\xi|^2$$

*for some  $\widehat{\kappa}_1, \widehat{\kappa}_2 > 0$  depending  $\kappa_1$  and  $\kappa_2$  and any symmetric matrix  $\xi = \{\xi_i^\alpha\}_{i,\alpha}$ .*

We assume  $A$  satisfies (1.2), (1.3) and (1.4). We assume  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain and  $\omega \subseteq \mathbb{R}^d$  is an unbounded Lipschitz domain with 1-periodic structure such that  $\mathbb{R}^d \setminus \omega$  is not connected but each connected component is separated by a positive distance  $\mathfrak{g}^\omega$ . We also assume that each connected component of  $\mathbb{R}^d \setminus \omega$  is bounded.

Let  $K_\varepsilon$  be defined as in Section 2. Let  $\eta_\varepsilon \in C_0^\infty(\Omega)$  satisfy

$$\begin{cases} 0 \leq \eta_\varepsilon(x) \leq 1 & \text{for } x \in \Omega, \\ \text{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 3\varepsilon\}, \\ \eta_\varepsilon = 1 & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 4\varepsilon\}, \\ |\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}. \end{cases} \quad (3.1)$$

If  $P_\varepsilon$  is the linear extension operator provided by Theorem 2.5, then we write  $\widetilde{w} = P_\varepsilon w$  for  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ . Throughout,  $C$  denotes a harmless constant that may change from line to line.

**Lemma 3.2.** *Let*

$$r_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_0)\eta_\varepsilon).$$

*Then*

$$\begin{aligned} & \int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w \\ &= |Q \cap \omega| \int_{\Omega} \widehat{A} \nabla u_0 \cdot \nabla \eta_\varepsilon \widetilde{w} - |Q \cap \omega| \int_{\Omega} (1 - \eta_\varepsilon) \widehat{A} \nabla u_0 \cdot \nabla \widetilde{w} \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^{\varepsilon} A^{\varepsilon} \right] \left[ \nabla u_0 - K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \right] \cdot \nabla \tilde{w} \\
& + \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^{\varepsilon} A^{\varepsilon} \nabla \chi^{\varepsilon} \right] K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla \tilde{w} \\
& - \varepsilon \int_{\Omega_{\varepsilon}} A^{\varepsilon} \chi^{\varepsilon} \nabla K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla w
\end{aligned}$$

for any  $w \in H^1(\Omega_{\varepsilon}, \Gamma_{\varepsilon}; \mathbb{R}^d)$ .

*Proof.* Since  $u_{\varepsilon}$  and  $u_0$  solve (1.5) and (2.6), respectively,

$$\int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla w = 0$$

and

$$|Q \cap \omega| \int_{\Omega} \hat{A} \nabla u_0 \cdot \nabla (\tilde{w} \eta_{\varepsilon}) = 0$$

for any  $w \in H^1(\Omega_{\varepsilon}, \Gamma_{\varepsilon}; \mathbb{R}^d)$ . Hence,

$$\begin{aligned}
& \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla r_{\varepsilon} \cdot \nabla w \\
& = \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla w - \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla u_0 \cdot \nabla w \\
& \quad - \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla [\varepsilon \chi^{\varepsilon} K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})] \cdot \nabla w \\
& = |Q \cap \omega| \int_{\Omega} \hat{A} \nabla u_0 \cdot \nabla (\tilde{w} \eta_{\varepsilon}) - \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla u_0 \cdot \nabla w \\
& \quad - \int_{\Omega_{\varepsilon}} A^{\varepsilon} \nabla \chi^{\varepsilon} K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla w \\
& \quad - \varepsilon \int_{\Omega_{\varepsilon}} A^{\varepsilon} \chi^{\varepsilon} \nabla K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla w \\
& = |Q \cap \omega| \int_{\Omega} \hat{A} \nabla u_0 \cdot \nabla \eta_{\varepsilon} \tilde{w} - |Q \cap \omega| \int_{\Omega} (1 - \eta_{\varepsilon}) \hat{A} \nabla u_0 \cdot \nabla \tilde{w} \\
& \quad + \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^{\varepsilon} A^{\varepsilon} \right] \left[ \nabla u_0 - K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \right] \cdot \nabla \tilde{w} \\
& \quad + \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^{\varepsilon} A^{\varepsilon} - \mathbf{1}_+^{\varepsilon} A^{\varepsilon} \nabla \chi^{\varepsilon} \right] K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla \tilde{w} \\
& \quad - \varepsilon \int_{\Omega_{\varepsilon}} A^{\varepsilon} \chi^{\varepsilon} \nabla K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon}) \cdot \nabla w,
\end{aligned}$$

which is the desired equality.  $\square$

Lemmas 3.3 presented below is used in the proof of Lemma 3.4, which establishes a Poincaré type inequality for the perforated domain. We use the notation  $\Delta(x, r) = B(x, r) \cap \partial\Omega$  to denote a surface ball of  $\partial\Omega$ .

**Lemma 3.3.** *For sufficiently small  $\varepsilon$ , there exist  $r_0, \rho_0 > 0$  depending only on  $\omega$  such that for any  $x \in \partial\Omega$ ,*

$$\Delta(y, \varepsilon\rho_0) \subset \Delta(x, \varepsilon r_0) \text{ and } \overline{\Delta(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$$

for some  $y \in \Gamma_\varepsilon$ .

*Proof.* Write  $\mathbb{R}^d \setminus \omega = \cup_{j=1}^\infty H_j$ , where each  $H_j$  is connected and bounded by assumption (see Section 2). Since  $\omega$  is 1-periodic, there exists a constant  $M < \infty$  such that

$$\sup_{j \geq 1} \{\text{diam } H_j\} \leq M.$$

Take

$$r_0 = 2 \max \{\mathfrak{g}^\omega, M\}, \quad (3.2)$$

where  $\mathfrak{g}^\omega$  is defined in Section 2. Set  $\rho_0 = \frac{1}{16}\mathfrak{g}^\omega$ . Let

$$\tilde{H}_j = \left\{ z \in \mathbb{R}^d : \text{dist}(z, H_j) < \frac{1}{4}\mathfrak{g}^\omega \right\} \text{ for each } j,$$

and fix  $x \in \partial\Omega$ . If  $x \in \partial\Omega \setminus (\cup_{j=1}^\infty \varepsilon\tilde{H}_j)$ , then take  $y = x$ . Indeed, for any  $z \in \Delta(y, \varepsilon\rho_0) \subset \Delta(x, \varepsilon r_0)$  and any positive integer  $k$ ,

$$\begin{aligned} \text{dist}(z, \varepsilon H_k) &\geq \text{dist}(y, \varepsilon H_k) - |y - z| \\ &\geq \varepsilon \frac{1}{4}\mathfrak{g}^\omega - \varepsilon\rho_0 \\ &\geq \varepsilon \left\{ \frac{1}{4}\mathfrak{g}^\omega - \frac{1}{16}\mathfrak{g}^\omega \right\} \\ &\geq \varepsilon \frac{3}{16}\mathfrak{g}^\omega, \end{aligned}$$

and so  $\overline{\Delta(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$ .

Suppose  $x \in \partial\Omega \cap (\cup_{j=1}^\infty \varepsilon\tilde{H}_j)$ . There exists a positive integer  $k$  such that  $x \in \varepsilon\tilde{H}_k$ . Moreover,  $\varepsilon\tilde{H}_k \subset B(x, \varepsilon r_0)$  since for any  $z \in \varepsilon\tilde{H}_k$  we have

$$|x - z| \leq \text{dist}(x, \varepsilon H_k) + \text{diam}(\varepsilon H_k) + \text{dist}(z, \varepsilon H_k)$$

$$\begin{aligned}
&\leq \varepsilon \frac{1}{4} \mathfrak{g}^\omega + \varepsilon M + \varepsilon \frac{1}{4} \mathfrak{g}^\omega \\
&< \varepsilon \mathfrak{g}^\omega + \varepsilon M \\
&< \varepsilon r_0.
\end{aligned}$$

In this case, choose  $y \in \varepsilon(\tilde{H}_k \setminus H_k)$  so that  $\text{dist}(y, \varepsilon H_k) = \varepsilon(1/8)\mathfrak{g}^\omega$  and  $y \in \partial\Omega$ . Then for any  $z \in \Delta(y, \varepsilon\rho_0) \subset [\partial\Omega \cap \varepsilon(\tilde{H}_k \setminus H_k)] \subset \Delta(x, \varepsilon r_0)$ ,

$$\begin{aligned}
\text{dist}(z, \varepsilon H_k) &\geq \text{dist}(y, \varepsilon H_k) - |y - z| \\
&\geq \varepsilon \frac{1}{8} \mathfrak{g}^\omega - \varepsilon \frac{1}{16} \mathfrak{g}^\omega \\
&\geq \varepsilon \frac{1}{16} \mathfrak{g}^\omega,
\end{aligned}$$

and so  $\overline{\Delta(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$ .  $\square$

**Lemma 3.4.** For  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ ,

$$\|\tilde{w}\|_{L^2(\mathcal{O}_{4\varepsilon})} \leq C\varepsilon \|\nabla \tilde{w}\|_{L^2(\Omega)},$$

where  $\mathcal{O}_{4\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < 4\varepsilon\}$  and  $C$  depends on  $d$ ,  $\Omega$ , and  $\omega$ .

*Proof.* We cover  $\partial\Omega$  with the surface balls  $\Delta(x, \varepsilon r_0)$  provided in Lemma 3.3 and partition the region  $\mathcal{O}_{4\varepsilon}$ . In particular, let  $r_0$  denote the constant given by Lemma 3.3, and note  $\cup_{x \in \partial\Omega} \Delta(x, \varepsilon r_0)$  covers  $\partial\Omega$ , which is compact. Then there exists  $\{x_i\}_{i=1}^N$  with  $\partial\Omega \subset \cup_{i=1}^N \Delta(x_i, \varepsilon r_0)$ , where  $N = N(\varepsilon)$ . Write

$$\mathcal{O}_{4\varepsilon}^{(i)} = \{x \in \Omega : \text{dist}(x, \Delta_i) < 4\varepsilon\}, \text{ where } \Delta_i = \Delta(x_i, \varepsilon r_0).$$

Given that  $\Omega$  is a Lipschitz domain, there exists a positive integer  $M < \infty$  independent of  $\varepsilon$  such that  $\mathcal{O}_{4\varepsilon}^{(i)} \cap \mathcal{O}_{4\varepsilon}^{(j)} \neq \emptyset$  for at most  $M$  positive integers  $j$  different from  $i$ .

Set  $W(x) = \tilde{w}(\varepsilon x)$ . Note for each  $1 \leq i \leq N$ , by Lemma 3.3 there exists a  $y_i \in \mathcal{O}_{4\varepsilon}^{(i)}$  such that  $\tilde{w} \equiv 0$  on  $\Delta(y_i, \varepsilon\rho_0) \subset \Delta_i$ . Hence, by Poincaré's inequality (see Theorem 1 in [9]),

$$\left( \int_{\mathcal{O}_{4\varepsilon}^{(i)}/\varepsilon} |W|^2 \right)^{1/2} \leq C \left( \int_{\mathcal{O}_{4\varepsilon}^{(i)}/\varepsilon} |\nabla W|^2 \right)^{1/2}, \quad (3.3)$$

where  $C$  depends on  $\Omega$ ,  $r_0$ , and  $\rho_0$  but is independent of  $\varepsilon$  and  $i$ . Specifically,

$$\int_{\mathcal{O}_{4\varepsilon}} |\tilde{w}(x)|^2 dx \leq C\varepsilon^2 \sum_{i=1}^N \int_{\mathcal{O}_{4\varepsilon}^{(i)}} |\nabla \tilde{w}(x)|^2 dx \leq C_1 \varepsilon^2 \int_{\mathcal{O}_{4\varepsilon}} |\nabla \tilde{w}(x)|^2 dx$$

where we've made the change of variables  $\varepsilon x \mapsto x$  in (3.3) and  $C_1$  is a constant depending on  $\Omega$ ,  $\omega$ , and  $M$  but independent of  $\varepsilon$ .  $\square$

**Lemma 3.5.** For  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ ,

$$\left| \int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w \right| \leq C \left\{ \|u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right. \\ \left. + \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|w\|_{H^1(\Omega_\varepsilon)}$$

*Proof.* By Lemma 3.2,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w = I_1 + I_2 + I_3 + I_4 + I_5, \quad (3.4)$$

where

$$\begin{aligned} I_1 &= |Q \cap \omega| \int_{\Omega} \hat{A} \nabla u_0 \cdot \nabla \eta_\varepsilon \tilde{w}, \\ I_2 &= -|Q \cap \omega| \int_{\Omega} (1 - \eta_\varepsilon) \hat{A} \nabla u_0 \cdot \nabla \tilde{w}, \\ I_3 &= \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon \right] [\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)] \cdot \nabla \tilde{w}, \\ I_4 &= \int_{\Omega} \left[ |Q \cap \omega| \hat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon \nabla \mathbb{X}^\varepsilon \right] K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla \tilde{w}, \\ I_5 &= -\varepsilon \int_{\Omega_\varepsilon} A^\varepsilon \chi^\varepsilon \nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w, \end{aligned}$$

and  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ . According to (3.1),  $\text{supp}(\nabla \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$ , where  $\mathcal{O}_{4\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < 4\varepsilon\}$ . Moreover,  $|\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}$ . Hence, Lemma 3.4, Lemma 3.1, and (3.1) imply

$$|I_1| \leq C\varepsilon^{-1} \int_{\mathcal{O}_{4\varepsilon}} |\nabla u_0 \cdot \tilde{w}| \leq C \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} \|\nabla \tilde{w}\|_{L^2(\Omega)}.$$

Since  $\text{supp}(1 - \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$  and  $\eta_\varepsilon \leq 1$ , Lemma 2.3 and Lemma 3.1 imply

$$|I_2| \leq C \int_{\mathcal{O}_{4\varepsilon}} \left| \hat{A} \nabla u_0 \cdot \nabla \tilde{w} \right| \leq C \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} \|\nabla \tilde{w}\|_{L^2(\Omega)}.$$

By Theorem 2.5,

$$|I_1 + I_2| \leq C \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} \|w\|_{H^1(\Omega_\varepsilon)}. \quad (3.5)$$

Again, since  $\text{supp}(1 - \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$  (see (3.1)),

$$\begin{aligned} & \|\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ & \leq \|(1 - \eta_\varepsilon)\nabla u_0\|_{L^2(\Omega)} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ & \quad + \|K_\varepsilon((\nabla u_0)\eta_\varepsilon) - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ & \leq \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} + C\|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_3| & \leq C \|\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \|w\|_{H^1(\Omega_\varepsilon)} \\ & \leq C \left\{ \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} \right. \\ & \quad \left. + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|w\|_{H^1(\Omega_\varepsilon)}. \end{aligned} \quad (3.6)$$

Set  $B = |Q \cap \omega| \hat{A} - \mathbf{1}_+ A \nabla \mathbb{X}$ . By (2.7) and (2.8),  $B$  satisfies the assumptions of Lemma 2.4. Therefore, there exists  $\pi = \{\pi_{kij}^{\alpha\beta}\}$  that is 1-periodic with

$$\frac{\partial}{\partial y_k} \pi_{kij}^{\alpha\beta} = b_{ij}^{\alpha\beta} \quad \text{and} \quad \pi_{kij}^{\alpha\beta} = -\pi_{ikj}^{\alpha\beta},$$

where

$$b_{ij}^{\alpha\beta} = |Q \cap \omega| \hat{a}_{ij}^{\alpha\beta} - \mathbf{1}_+ a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} \mathbb{X}_j^{\gamma\beta}.$$

Moreover,  $\|\pi_{ij}^{\alpha\beta}\|_{H^1(Q)} \leq C$  for some constant  $C$  depending on  $\kappa_1$ ,  $\kappa_2$ , and  $\omega$ . Hence, integrating by parts gives

$$\begin{aligned} \int_{\Omega} b_{ij}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \frac{\partial \tilde{w}^\alpha}{\partial x_i} & = -\varepsilon \int_{\Omega} \pi_{kij}^{\alpha\beta\varepsilon} \frac{\partial}{\partial x_k} \left[ K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \frac{\partial \tilde{w}^\alpha}{\partial x_i} \right] \\ & = -\varepsilon \int_{\Omega} \pi_{kij}^{\alpha\beta\varepsilon} \frac{\partial}{\partial x_k} \left[ K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \right] \frac{\partial \tilde{w}^\alpha}{\partial x_i}, \end{aligned}$$

since

$$\int_{\Omega} \pi_{kij}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \frac{\partial^2 \tilde{w}^\alpha}{\partial x_k \partial x_i} = 0$$

due to the anti-symmetry of  $\pi$ . Thus, by Lemma 2.2, and (3.1),

$$|I_4| \leq C\varepsilon \|\pi^\varepsilon \nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \|w\|_{H^1(\Omega_\varepsilon)}$$

$$\leq C \left\{ \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|w\|_{H^1(\Omega_\varepsilon)}. \quad (3.7)$$

Finally, by Lemma 2.2, and (3.1),

$$|I_5| \leq C \left\{ \|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|w\|_{H^1(\Omega_\varepsilon)} \quad (3.8)$$

The desired estimate follows from (3.4), (3.5), (3.6), (3.7), and (3.8).  $\square$

**Lemma 3.6.** *For  $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ ,*

$$\left| \int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w \right| \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)} \|w\|_{H^1(\Omega_\varepsilon)}$$

*Proof.* Recall that  $u_0$  satisfies  $\mathcal{L}_0(u_0) = 0$  in  $\Omega$ , and so it follows from estimates for solutions in Lipschitz domains for constant-coefficient equations that

$$\|(\nabla u_0)^*\|_{L^2(\partial\Omega)} \leq C \|f\|_{H^1(\partial\Omega)}, \quad (3.9)$$

where  $(\nabla u_0)^*$  denotes the nontangential maximal function of  $\nabla u_0$  (see [7]). By the coarea formula,

$$\|\nabla u_0\|_{L^2(\mathcal{O}_{4\varepsilon})} \leq C \varepsilon^{1/2} \|(\nabla u_0)^*\|_{L^2(\partial\Omega)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)}. \quad (3.10)$$

Notice that if  $u_0$  solves (2.6), then  $\mathcal{L}_0(\nabla u_0) = 0$  in  $\Omega$ , and so we may use the interior estimate for  $\mathcal{L}_0$ . That is,

$$|\nabla^2 u_0(x)| \leq \frac{C}{\delta(x)} \left( \int_{B(x, \delta(x)/8)} |\nabla u_0|^2 \right)^{1/2}, \quad (3.11)$$

where  $\delta(x) = \text{dist}(x, \partial\Omega)$ . In particular,

$$\begin{aligned} \|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)} &\leq \left( \int_{\Omega \setminus \mathcal{O}_{3\varepsilon}} |\nabla^2 u_0|^2 \right)^{1/2} \\ &\leq C \left( \int_{\Omega \setminus \mathcal{O}_{3\varepsilon}} \int_{B(x, \delta(x)/8)} \left| \frac{\nabla u_0(y)}{\delta(x)} \right|^2 dy dx \right)^{1/2} \\ &\leq C \left( \int_{3\varepsilon}^{C_0} t^{-2} \int_{\partial\mathcal{O}_t \cap \Omega} \int_{B(x, t/8)} |\nabla u_0(y)|^2 dy dS(x) dt \right)^{1/2} \\ &\quad + C_1 \left( \int_{\Omega \setminus \mathcal{O}_{C_0}} |\nabla u_0|^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq C \|(\nabla u_0)^*\|_{L^2(\partial\Omega)} \left( \int_{3\varepsilon}^{C_0} t^{-2} dt \right)^{1/2} + C_1 \|\nabla u_0\|_{L^2(\Omega)} \\
&\leq C \left\{ \varepsilon^{-1/2} \|f\|_{H^1(\partial\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)} \right\} \\
&\leq C \varepsilon^{-1/2} \|f\|_{H^1(\partial\Omega)}.
\end{aligned} \tag{3.12}$$

where  $C_0$  is a constant depending on  $\Omega$ , and we've used (3.1), (3.11), the coarea formula, energy estimates, and (3.9). Hence,

$$\varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)}. \tag{3.13}$$

Finally, by Lemma 2.1,

$$\|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)}. \tag{3.14}$$

where the last inequality follows from (3.1), Lemma 2.1, and (3.12). Equations (3.10), (3.13), and (3.14) together with Lemma 3.5 give the desired estimate.  $\square$

*Proof of Theorem 1.4.* Note  $r_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ , and so by Lemma 3.6 and (1.3),

$$\begin{aligned}
\|e(r_\varepsilon)\|_{L^2(\Omega_\varepsilon)}^2 &\leq C \int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla r_\varepsilon \\
&\leq C \varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)} \|r_\varepsilon\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Lemma 2.6 gives the desired estimate.  $\square$

## 4 Interior Lipschitz Estimate

In this section, we use Theorem 1.4 to investigate interior Lipschitz estimates down to the scale  $\varepsilon$ . In particular, we prove Theorem 1.1. The proof of Theorem 1.1 is based on the scheme used in [11] to prove boundary Lipschitz estimates for solutions to (1.5) in the case  $\omega = \mathbb{R}^d$ , which in turn is based on a more general scheme for establishing Lipschitz estimates presented in [3] and adapted in [11] and [2].

The following Lemma is essentially Cacciopoli's inequality in a perforated ball. The proof is similar to a proof of the classical Cacciopoli's inequality, but nevertheless we present a proof for completeness.

Throughout this section, let  $B_\varepsilon(r)$  denote the perforated ball of radius  $r$  centered at some  $x_0 \in \mathbb{R}^d$ , i.e.,  $B_\varepsilon(r) = B(x_0, r) \cap \varepsilon\omega$ . Let  $S_\varepsilon(r) = \partial(\varepsilon\omega) \cap B(x_0, r)$  and  $\Gamma_\varepsilon(r) = \varepsilon\omega \cap \partial B(x_0, r)$ .



**Lemma 4.1.** *Suppose  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B_\varepsilon(2)$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  on  $S_\varepsilon(2)$ . There exists a constant  $C$  depending on  $\kappa_1$  and  $\kappa_2$  such that*

$$\left( \int_{B_\varepsilon(1)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(2)} |u_\varepsilon - q|^2 \right)^{1/2}$$

*Proof.* Let  $\varphi \in C_0^\infty(B(2))$  satisfy  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $B(1)$ ,  $|\nabla \varphi| \leq C_1$  for some constant  $C_1$ . Let  $q \in \mathbb{R}^d$ , and set  $w = (u_\varepsilon - q)\varphi^2$ . By (1.1) and Hölder's inequality,

$$\begin{aligned} 0 &= \int_{B_\varepsilon(2)} A^\varepsilon \nabla u_\varepsilon \nabla w \\ &\geq C_2 \int_{B_\varepsilon(2)} |e(u_\varepsilon)|^2 \varphi^2 - C_3 \int_{B_\varepsilon(2)} |\nabla \varphi|^2 |u_\varepsilon - q|^2 \end{aligned} \quad (4.1)$$

for some constants  $C_2$  and  $C_3$  depending on  $\kappa_1$  and  $\kappa_2$ . In particular,

$$\int_{B_\varepsilon(2)} |e(u_\varepsilon \varphi)|^2 \leq C \int_{B_\varepsilon(2)} |\nabla \varphi|^2 |u_\varepsilon - q|^2,$$

where  $C$  only depends on  $\kappa_1$  and  $\kappa_2$ . Since  $\varphi \equiv 1$  in  $B(1)$  and  $u_\varepsilon \varphi \in H^1(B_\varepsilon(2), \Gamma_\varepsilon(2); \mathbb{R}^d)$ , equation (4.1) together with Lemma 2.6 gives the desired estimate.  $\square$

We extend Lemma 4.1 to hold for a ball  $B_\varepsilon(r)$  with  $r > 0$  by a convenient scaling technique—the so called “blow-up argument”—often used in the study of homogenization.

**Lemma 4.2.** *Suppose  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B_\varepsilon(2r)$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  on  $S_\varepsilon(2r)$ . There exists a constant  $C$  depending on  $\kappa_1$  and  $\kappa_2$  such that*

$$\left( \int_{B_\varepsilon(r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(2r)} |u_\varepsilon - q|^2 \right)^{1/2}$$

*Proof.* Let  $U_\varepsilon(x) = u_\varepsilon(rx)$ , and note  $U_\varepsilon$  satisfies  $\mathcal{L}_{\varepsilon/r}(U_\varepsilon) = 0$  in  $B_\varepsilon(2)$  and  $\sigma_{\varepsilon/r}(U_\varepsilon) = 0$  on  $S_\varepsilon(2)$ . By Lemma 4.1,

$$\left( \int_{B_{\varepsilon/r}(1)} |\nabla U_\varepsilon|^2 \right)^{1/2} \leq C \inf_{q \in \mathbb{R}^d} \left( \int_{B_{\varepsilon/r}(2)} |U_\varepsilon - q|^2 \right)^{1/2}$$

for some  $C$  independent of  $\varepsilon$  and  $r$ . Note  $\nabla U_\varepsilon = r \nabla u_\varepsilon$ , and so

$$r^{1-d/2} \left( \int_{B_\varepsilon(r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C r^{-d/2} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(2r)} |u_\varepsilon - q|^2 \right)^{1/2},$$

where we've made the substitution  $rx \mapsto x$ . The desired inequality follows.  $\square$

The following lemma is a key estimate in the proof of Theorem 1.1. Intrinsically, the following Lemma uses the convergence rate in Theorem 1.4 to approximate the solution  $u_\varepsilon$  with a “nice” function.

**Lemma 4.3.** *Suppose  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B_\varepsilon(3r)$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  on  $S_\varepsilon(3r)$ . There exists a  $v \in H^1(B(r); \mathbb{R}^d)$  with  $\mathcal{L}_0(v) = 0$  in  $B(r)$  and*

$$\left( \int_{B_\varepsilon(r)} |u_\varepsilon - v| \right)^{1/2} \leq C \left( \frac{\varepsilon}{r} \right)^{1/2} \left( \int_{B_\varepsilon(3r)} |u_\varepsilon|^2 \right)^{1/2}$$

for some constant  $C$  depending on  $d, \omega, \kappa_1$ , and  $\kappa_2$

*Proof.* With rescaling (see the proof of Lemma 4.2), we may assume  $r = 1$ . By Lemma 4.2 and estimate (2.3) of Lemma 2.5,

$$\left( \int_{B(3/2)} |\tilde{u}_\varepsilon|^2 \right)^{1/2} + \left( \int_{B(3/2)} |\nabla \tilde{u}_\varepsilon|^2 \right)^{1/2} \leq C \left( \int_{B_\varepsilon(3)} |u_\varepsilon|^2 \right)^{1/2},$$

where  $\tilde{u}_\varepsilon = P_\varepsilon u_\varepsilon \in H^1(B(3); \mathbb{R}^d)$  and  $P_\varepsilon$  is the linear extension operator provided in Lemma 2.5. The coarea formula then implies there exists a  $t \in [1, 3/2]$  such that

$$\|\nabla \tilde{u}_\varepsilon\|_{L^2(\partial B(t))} + \|\tilde{u}_\varepsilon\|_{L^2(\partial B(t))} \leq C \|u_\varepsilon\|_{L^2(B_\varepsilon(3))}. \quad (4.2)$$

Let  $v$  denote the solution to the Dirichlet problem  $\mathcal{L}_0(v) = 0$  in  $B(t)$  and  $v = \tilde{u}_\varepsilon$  on  $\partial B(t)$ . Note that  $v = u_\varepsilon = \tilde{u}_\varepsilon$  on  $\Gamma_\varepsilon(t)$ . By Theorem 1.4,

$$\|u_\varepsilon - v\|_{L^2(B_\varepsilon(t))} \leq C\varepsilon^{1/2} \|\tilde{u}_\varepsilon\|_{H^1(\partial B(t))}$$

since

$$\|\chi^\varepsilon K_\varepsilon^2((\nabla v)\eta_\varepsilon)\|_{L^2(B_\varepsilon(t))} \leq C \|\nabla v\|_{L^2(B(t))},$$

where we've used notation consistent with Theorem 1.4. Hence, (4.2) gives

$$\|u_\varepsilon - v\|_{L^2(B_\varepsilon(1))} \leq \|u_\varepsilon - v\|_{L^2(B_\varepsilon(t))} \leq C\varepsilon^{1/2} \|u_\varepsilon\|_{L^2(B_\varepsilon(3))}.$$

$\square$

**Lemma 4.4.** *Suppose  $\mathcal{L}_0(v) = 0$  in  $B(2r)$ . For  $r \geq \varepsilon$ , there exists a constant  $C$  depending on  $\omega, \kappa_1, \kappa_2$  and  $d$  such that*

$$\left( \int_{B(r)} |v|^2 \right)^{1/2} \leq C \left( \int_{B_\varepsilon(2r)} |v|^2 \right)^{1/2} \quad (4.3)$$

*Proof.* Let

$$T_\varepsilon = \{z \in \mathbb{Z}^d : \varepsilon(Q+z) \cap B(r) \neq \emptyset\},$$

and fix  $z \in T_\varepsilon$ . Let  $\{H_k\}_{k=1}^N$  denote the bounded, connected components of  $\mathbb{R}^d \setminus \omega$  with  $H_k \cap (Q+z) \neq \emptyset$ . Define  $\varphi_k \in C_0^\infty(Q^*(z))$  by

$$\begin{cases} \varphi_k(x) = 1, & \text{if } x \in H_k, \\ \varphi_k(x) = 0, & \text{if } \text{dist}(x, H_k) > \frac{1}{4}\mathfrak{g}^\omega, \\ |\nabla \varphi_k| \leq C, \end{cases}$$

where  $C$  depends on  $\omega$ ,  $\mathfrak{g}^\omega > 0$  is defined in Section 2 by (2.9), and

$$Q^*(z) = \bigcup_{j=1}^{3^d} (Q + z_j), \quad z_j \in \mathbb{Z}^d \text{ and } |z - z_j| \leq \sqrt{d}.$$

Set  $\varphi = \sum_{k=1}^N \varphi_k \in C_0^\infty(Q^*)$ , where  $Q^* = Q^*(z)$ . Note by construction  $\varphi \equiv 1$  in  $Q^* \setminus \omega$ .

Set  $V(x) = v(\varepsilon x)$ . Note  $\mathcal{L}_0(V) = 0$  in  $Q + z$ . By Poincaré's and Caccioppoli's inequalities,

$$\int_{(Q+z) \setminus \omega} |V|^2 \leq \sum_{k=1}^N \int_{H_k} |V|^2 \leq C \int_{Q^*} |\nabla(V\varphi)|^2 \leq C \int_{Q^*} |V|^2 |\nabla \varphi|^2,$$

where  $C$  depends on  $\omega, \kappa_1, \kappa_2$ , and  $d$  but is independent of  $z$ . Specifically, since  $\nabla \varphi = 0$  in  $Q^* \setminus \omega$  and  $(Q+z) \subset Q^*$ ,

$$\int_{(Q+z) \cap \omega} |V|^2 + \int_{(Q+z) \setminus \omega} |V|^2 \leq C \int_{Q^* \cap \omega} |V|^2,$$

where  $C$  only depends on  $\omega, \kappa_1, \kappa_2$ , and  $d$ . Making the change of variables  $\varepsilon x \mapsto x$  gives

$$\int_{\varepsilon(Q+z)} |v|^2 \leq C \int_{\varepsilon(Q^* \cap \omega)} |v|^2.$$

Summing over all  $z \in T_\varepsilon$  gives the desired inequality, since there is a constant  $M < \infty$  depending only on  $d$  such that  $Q^*(z_1) \cap Q^*(z_2) \neq \emptyset$  for at most  $M$  coordinates  $z_2 \in \mathbb{Z}^d$  different from  $z_1$ .  $\square$

For  $w \in L^2(B_\varepsilon(r); \mathbb{R}^d)$  and  $\varepsilon, r > 0$ , set

$$H_\varepsilon(r; w) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} \left( \int_{B_\varepsilon(r)} |w - Mx - q|^2 \right)^{1/2}, \quad (4.4)$$

and set

$$H_0(r; w) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} \left( \int_{B(r)} |w - Mx - q|^2 \right)^{1/2}.$$

**Lemma 4.5.** *Let  $v$  be a solution of  $\mathcal{L}_0(v) = 0$  in  $B(r)$ . For  $r \geq \varepsilon$ , there exists a  $\theta \in (0, 1/4)$  such that*

$$H_\varepsilon(\theta r; v) \leq \frac{1}{2} H_\varepsilon(r; v).$$

*Proof.* There exists a constant  $C_1$  depending on  $d$  such that

$$H_\varepsilon(r; v) \leq C_1 H_0(r; v)$$

for any  $r > 0$ . It follows from interior  $C^2$ -estimates for elasticity systems with constant coefficients that there exists  $\theta \in (0, 1/4)$  with

$$H_0(\theta r; v) \leq \frac{1}{2C_2} H_0(r/2; v),$$

where  $C_2 = C_3 C_1$  and  $C_3$  is the constant in (4.3) given in Lemma 4.4. By Lemma 4.4, we have the desired inequality.  $\square$

**Lemma 4.6.** *Suppose  $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$  in  $B_\varepsilon(2r)$  and  $\sigma_\varepsilon(u_\varepsilon) = 0$  on  $S_\varepsilon(2r)$ . For  $r \geq \varepsilon$ ,*

$$H_\varepsilon(\theta r; u_\varepsilon) \leq \frac{1}{2} H_\varepsilon(r; u_\varepsilon) + \frac{C}{r} \left( \frac{\varepsilon}{r} \right)^{1/2} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(3r)} |u_\varepsilon - q|^2 \right)^{1/2}$$

*Proof.* With  $r$  fixed, let  $v_r \equiv v$  denote the function guaranteed in Lemma 4.3. Observe then

$$\begin{aligned} H_\varepsilon(\theta r; u_\varepsilon) &\leq \frac{1}{\theta r} \left( \int_{B_\varepsilon(\theta r)} |u_\varepsilon - v|^2 \right)^{1/2} + H_\varepsilon(\theta r; v) \\ &\leq \frac{C}{r} \left( \int_{B_\varepsilon(r)} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; v) \end{aligned}$$

$$\leq \frac{C}{r} \left( \int_{B_\varepsilon(r)} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; u_\varepsilon),$$

where we've used Lemma 4.5. By Lemma 4.3, we have

$$H_\varepsilon(\theta r; u_\varepsilon) \leq \frac{C}{r} \left( \frac{\varepsilon}{r} \right)^{1/2} \left( \int_{B_\varepsilon(3r)} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; u_\varepsilon).$$

Since  $H$  remains invariant if we subtract a constant from  $u_\varepsilon$ , the desired inequality follows.  $\square$

**Lemma 4.7.** *Let  $H(r)$  and  $h(r)$  be two nonnegative continuous functions on the interval  $(0, 1]$ . Let  $0 < \varepsilon < 1/6$ . Suppose that there exists a constant  $C_0$  with*

$$\begin{cases} \max_{r \leq t \leq 3r} H(t) \leq C_0 H(3r), \\ \max_{r \leq t, s \leq 3r} |h(t) - h(s)| \leq C_0 H(3r), \end{cases}$$

for any  $r \in [\varepsilon, 1/3]$ . We further assume

$$H(\theta r) \leq \frac{1}{2} H(r) + C_0 \left( \frac{\varepsilon}{r} \right)^{1/2} \{H(3r) + h(3r)\}$$

for any  $r \in [\varepsilon, 1/3]$ , where  $\theta \in (0, 1/4)$ . Then

$$\max_{\varepsilon \leq r \leq 1} \{H(r) + h(r)\} \leq C \{H(1) + h(1)\},$$

where  $C$  depends on  $C_0$  and  $\theta$ .

*Proof.* See [11].  $\square$

*Proof of Theorem 1.1.* By rescaling, we may assume  $R = 1$ . We assume  $\varepsilon \in (0, 1/6)$ , and we let  $H(r) \equiv H_\varepsilon(r; u_\varepsilon)$ , where  $H_\varepsilon(r; u_\varepsilon)$  is defined above by (4.4). Let  $h(r) = |M_r|$ , where  $M_r \in \mathbb{R}^{d \times d}$  satisfies

$$H(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(r)} |u_\varepsilon - M_r x - q|^2 \right)^{1/2}.$$

Note there exists a constant  $C$  independent of  $r$  so that

$$H(t) \leq C H(3r), \quad t \in [r, 3r]. \quad (4.5)$$

Suppose  $s, t \in [r, 3r]$ . We have

$$\begin{aligned}
|h(t) - h(s)| &\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(r)} |(M_t - M_s)x - q|^2 \right)^{1/2} \\
&\leq \frac{C}{t} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(t)} |u_\varepsilon - M_t x - q|^2 \right)^{1/2} \\
&\quad + \frac{C}{s} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(s)} |u_\varepsilon - M_s x - q|^2 \right)^{1/2} \\
&\leq CH(3r),
\end{aligned}$$

where we've used (4.5) for the last inequality. Specifically,

$$\max_{r \leq t, s \leq 3r} |h(t) - h(s)| \leq CH(3r). \quad (4.6)$$

Clearly

$$\frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(3r)} |u_\varepsilon - q|^2 \right)^{1/2} \leq H(3r) + h(3r),$$

and so Lemma 4.6 implies

$$H(\theta r) \leq \frac{1}{2}H(r) + C \left( \frac{\varepsilon}{r} \right)^{1/2} \{H(3r) + h(3r)\} \quad (4.7)$$

for any  $r \in [\varepsilon, 1/3]$  and some  $\theta \in (0, 1/4)$ . Note equations (4.5), (4.6), and (4.7) show that  $H(r)$  and  $h(r)$  satisfy the assumptions of Lemma 4.7. Consequently,

$$\begin{aligned}
\left( \int_{B_\varepsilon(r)} |\nabla u_\varepsilon|^2 \right)^{1/2} &\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_\varepsilon(3r)} |u_\varepsilon - q|^2 \right)^{1/2} \\
&\leq C \{H(3r) + h(3r)\} \\
&\leq C \{H(1) + h(1)\} \\
&\leq C \left( \int_{B_\varepsilon(1)} |u_\varepsilon|^2 \right)^{1/2}. \quad (4.8)
\end{aligned}$$

Since (4.8) remains invariant if we subtract a constant from  $u_\varepsilon$ , the desired estimate in Theorem 1.1 follows.  $\square$

*Proof of Corollary 1.2.* Under the Hölder continuous condition (1.8) and the assumption that  $\omega$  is an unbounded  $C^{1,\alpha}$  domain for some  $\alpha > 0$ , solutions to the systems of linear elasticity are known to be locally Lipschitz. That is, if  $\mathcal{L}_1(u) = 0$  in  $B(y, 1) \cap \omega$  and  $\sigma_1(u) = 0$  on  $B(y, 1) \cap \partial\omega$ , then

$$\|\nabla u\|_{L^\infty(B(y, 1/3) \cap \omega)} \leq C \left( \int_{B(y, 1) \cap \omega} |\nabla u|^2 \right)^{1/2}, \quad (4.9)$$

where  $C$  depends on  $d$ ,  $\kappa_1$ ,  $\kappa_2$ , and  $\omega$ .

By rescaling, we may assume  $R = 1$ . To prove the desired estimate, assume  $\varepsilon \in (0, 1/6)$ . Indeed, if  $\varepsilon \geq 1/6$ , then (1.9) follows from (4.9). From (4.9), a “blow-up argument” (see the proof of Lemma 4.1), and Theorem 1.1 we deduce

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(B(y, \varepsilon) \cap \varepsilon\omega)} &\leq C \left( \int_{B(y, 3\varepsilon) \cap \varepsilon\omega} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\leq C \left( \int_{B(x_0, 1) \cap \varepsilon\omega} |\nabla u_\varepsilon|^2 \right)^{1/2} \end{aligned}$$

for any  $y \in B(x_0, 1/3)$ . The desired estimate readily follows by covering  $B(x_0, 1/3)$  with balls  $B(y, \varepsilon)$ .  $\square$

*Proof of Corollary 1.3.* If  $u$  satisfies the growth condition (1.10), then by Lemma 4.2 and Theorem 1.1,

$$\left( \int_{B(x_0, r) \cap \omega} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{B(x_0, R) \cap \omega} |\nabla u|^2 \right)^{1/2} \leq CR^{\nu-1},$$

where  $C$  is independent of  $R$ . Take  $R \rightarrow \infty$  and note  $\nabla u = 0$  for arbitrarily large  $r$ . Since  $\omega$  is connected, we conclude  $u$  is constant.  $\square$

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